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Existence and computation of spherical rational quartic curves for Hermite interpolation

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We study the existence and computation of spherical rational quartic curves that interpolate Hermite data on a sphere, i.e. two distinct endpoints and tangent vectors at the two points. It is shown that spherical rational quartic curves interpolating such data always exist, and that the family of these curves has n degrees of freedom for any given Hermite data on S^n, $n \geq 2$. A method is presented for generating all spherical rational quartic curves on S^n interpolating given Hermite data.

Key words: Spherical rational guartic curves – Hermite interpolation – Stereographic projection

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1 Introduction

Let X_0 and X_1 be two distinct points on the unit sphere $S^n \subset E^{n+1}$, $n \geq 2$. Let T_0 and T_1 be two nonzero vectors that are tangent to S^n at X_0 and X_1, respectively. Let $D = \{X_0, T_0; X_1, T_1\}$. We consider the problem of using a parametric curve $P(t)$, $t \in [0, 1]$, on S^n to interpolate D. In other words, we wish to find a curve $P(t)$, $t \in [0, 1]$, on S^n such that $P(0) = X_0$, $P(1) = X_1$, $P'(0) = T_0$, and $P'(1) = T_1$. For brevity, a curve that lies on a sphere is termed a *spherical curve*. The particular problems this paper is concerned with are the existence and computation of spherical rational (SR) quartic curves interpolating $D = \{X_0, T_0; X_1, T_1\}$ on S^n.

SR curves have only even degrees. The simplest SR curves are of degree 2, i.e. circles. Circular arcs have been used for data interpolation and approximation on a sphere in the form of circular arc splines or bi-arcs. General SR curves of degree $2d$ have been constructed in the literature as the images of rational curves of degree d under stereographic projection. However, no existing work addresses the problem of using SR quartic curves for Hermite interpolation with the data $D = \{X_0, T_0; X_1, T_1\}$ in general positions.

The main results of this paper are the following. It is shown that, for any data $D = \{X_0, T_0; X_1, T_1\}$ given on S^n, there exist SR quartic curves on S^n interpolating D, and all these curves form a family with n degrees of freedom. In addition, a method is presented to compute all SR quartic curves that interpolate D. This method is based on direct algebraic manipulation, instead of stereographic projection, which is used in most other existing methods for constructing general SR curves. In fact, we show that stereographic projection cannot generate all SR quartic curves on S^n as the images of rational quadratic curves when $n \geq 3$.

1.1 Related work

Curves in the unit quaternion space are used for modeling rotations in computer animation [8, 9]. These curves are essentially spherical curves, since the space of unit quaternions can be identified with S^3.

Among all spherical curves, the SR curves have simple expressions and are relatively easy to construct. The stereographic projection is used in [4] to construct general SR curves for interpolation on S^2. This approach consists of three main steps: (1) map interpolation data on S^2 to be interpolated into data in a 2D plane, (2) construct a planar rational curve of degree d to interpolate the mapped data in the plane and (3) map the planar rational curve back into an SR curve of degree $2d$ on S^2 to interpolate the original data. So far, most existing work on constructing SR curves has been based on the stereographic projection [1, 6, 14].

SR curves can only have even degrees. The SR curves of degree 2 are circular arcs. The construction of circular arc spline curves on a sphere has been discussed in [4, 11, 13]. An SR curve of degree 6 is constructed in [6] for interpolating Hermite data in the unit quaternion space, using a transformation from number theory, which is equivalent to the standard stereographic projection on S^3. A generalized form of stereographic projection is used in [14] to construct SR curves of degree 6 for Hermite interpolation. A different generalization of stereographic projection is studied in [7] for converting the problem of interpolation points on S^2 by SR curves into one of interpolating lines in 3-D space.

The obvious gap between SR quadratic curves and SR curves of degree 6 are the SR quartic curves. The use of SR quartic curves for Hermite interpolation has not been addressed in the literature. Only spherical quartic curves interpolating five data points on a sphere are considered in the study by Gfrerrer [3] on general rational interpolation on a hypersphere. This status is probably due to two limitations of applying stereographic projection. First, in a typical approach employing stereographic projection, SR quartic curves would have to be obtained as the images of rational quadratic curves. However, rational quadratic curves, as conic sections, have no inflection points, so they cannot interpolate general Hermite data. Second, although stereographic projection maps rational quadratic curves into SR quartic curves, not all SR quartic curves can be obtained in this way, even by using different centers of projection. In fact, we will show that, when $n \geq 3$, stereographic projection is incapable of generating all SR quartic curves as the images of rational quadratic curves on S^n. This motivates us to find a method that can generate all SR quartic curves on S^n interpolating Hermite data.

The rest of the paper is organized as follows. In Sect. 2 we prove the existence of SR quartic curves for Hermite interpolation. In Sect. 3 we present an algebraic method of computing all SR quartic curves interpolating given Hermite data. We show that the family of SR quartic curves interpolating given Hermite data on S^n has n degrees of freedom. The paper concludes in Sect. 4.

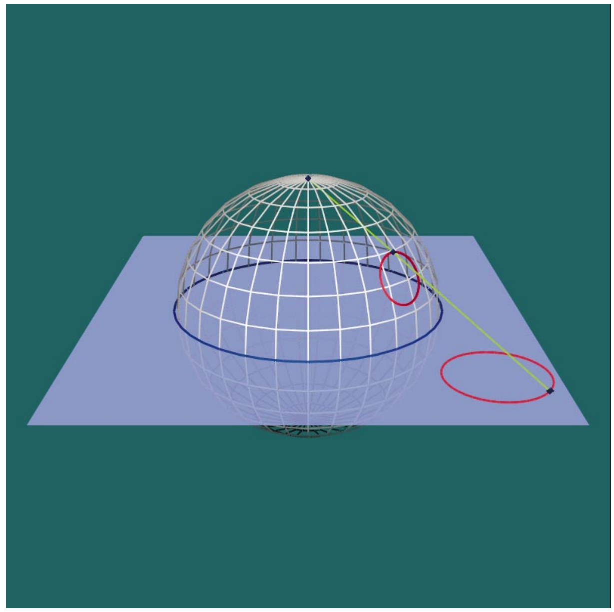

Fig. 1. Standard stereographic projection

2 Existence

Given data $D = \{X_0, T_0; X_1, T_1\}$ on a sphere S^n, $n \geq 2$, we show that there always exist SR quartic curves interpolating D. We further point out that not all of these curves can be obtained by stereographic projection unless $n = 2$.

The line determined by two points Y_0 and Y_1 is denoted by $[Y_0Y_1]$. The line determined by a point Y_0 and a directional vector U_0 is denoted by $[Y_0U_0]$. The standard stereographic projection is defined between S^2 and the plane $z = 0$ through a projection with its center at the north pole of S^2, i.e. $N = (0, 0, 1)$. See Fig. 1. We need to extend this definition in three ways. First, any point $C = (c_0, c_1, c_2)$ on S^2 can be used as the center of a stereographic projection. In this case, unless otherwise specified, the corresponding projection plane M_C passes through the origin and has (c_0, c_1, c_2) as its normal vector. Second, any plane not passing through the center of projection can be used as the projection plane. In this case, the stereographic projection is still birational, but in general no longer possesses the circle-preserving property. Third, we extend stereographic projection to a hypersphere S^n. In this case, the center of projection is a point C on S^n, and the mapping is defined between S^n and a hyperplane in E^{n+1}.

For general data $D = \{X_0, T_0; X_1, T_1\}$ on S^n, there exists an affine 3-space H that is spanned by points X_0 and X_1 and vectors T_0 and T_1; if the three vectors $X_1 - X_0$, T_0, and T_1 are linearly independent, H is a translation of the 3-D linear space spanned by $X_1 - X_0$, T_0, and T_1. Let S denote the 2-D sphere that is the intersection of S^n and H. Then we can consider a similar problem of interpolating $D = \{X_0, T_0; X_1, T_1\}$ by an SR quartic curve on S. Since the existence of solutions is invariant under scaling transformation, without loss of generality, we can replace S by S^2. Thus, we only need to prove that, for any data $D = \{X_0, T_0; X_1, T_1\}$ on S^2, there exist SR quartic curves on S^2 that interpolate D. Consequently, the existence proof for a hypersphere S^n will follow.

Theorem 1. *Let X_0 and X_1 be two distinct points on S^2. Let T_0 and T_1 be two nonzero vectors that are tangent to S^2 at X_0 and X_1, respectively. There exist SR quartic curves on S^2 that interpolate $D = \{X_0, T_0; X_1, T_1\}$.*

Proof. We use stereographic projection as the main mechanism in the proof. Consider the pencil of planes passing through $[X_0X_1]$. There are two planes P_0 and P_1 in this pencil that contain the tangent lines $[X_0T_0]$ and $[X_1T_1]$ of S^2, respectively. Now choose a point $C \in S^2$ distinct from X_0 and X_1 such that the plane determined by C, X_0 and X_1 is distinct from P_0 and P_1. Obviously, all points on S^2 can be selected as C, except those on the two circles $P_0 \cap S^2$ and $P_1 \cap S^2$.

Now consider the stereographic projection $\mathcal{P}_C$ centered at C from the projection plane M_C to S^2. According to the way C is chosen, X_0 and X_1 and the two tangent lines $[X_0T_0]$ and $[X_1T_1]$ are mapped by $\mathcal{P}_C^{-1}$ into two points Y_0 and Y_1 and two lines g and h, respectively, on the plane M_C, in such a configuration that the line g does not contain Y_1 and the line h does not contain Y_0. The four different configurations of the mapped data on plane M_C are shown in Fig. 2

Let $Z = g \cap h$ denote the intersection point between lines g and h. A key assumption we have to make is that Y_0, Y_1, and Z are finite points. This can be satisfied by properly choosing the projection plane M_C. Let W_i denote the plane determined by point C and line $[X_iT_i]$, $i = 0, 1$. Let L denote the intersection line between planes W_0 and W_1. Then Y_0, Y_1, and Z are all finite points on M_C as long as M_C is not parallel to any of the lines $[X_0C]$, $[X_1C]$, and L.

For a point Y in the plane M_C, the map $\mathcal{P}_C$ induces a linear map $\mathcal{T}_Y$ from the space of vectors originating at Y on the plane M_C to the space of tangent vectors to S^2 at $\mathcal{P}_C(Y)$. Let $U_0 = \mathcal{T}_{Y_0}^{-1}(T_0)$ and $U_1 = \mathcal{T}_{Y_1}^{-1}(T_1)$. Let $\hat{D} = \{Y_0, U_0; Y_1, U_1\}$. Clearly, if we can find a curve $Q(t)$ interpolating $\hat{D}$ in the plane M_C, then $P(t) = \mathcal{P}_C(Q(t))$ will be a spherical curve on S^2 interpolating D.

Now consider a rational quadratic Bézier curve

$$Q(t) = \frac{Q_0w_0B_{2,0}(t) + Q_1w_1B_{2,1}(t) + Q_2w_2B_{2,2}(t)}{w_0B_{2,0}(t) + w_1B_{2,1}(t) + w_2B_{2,2}(t)}, \quad t \in [0, 1]$$

with control points $Q_0 = Y_0$, $Q_2 = Y_1$, and $Q_1 = Z = g \cap h$. Since

$$Q'(0) = \frac{2w_0}{w_1}(Q_1 - Q_0) \text{ and } Q'(1) = \frac{2w_2}{w_1}(Q_2 - Q_1)$$

to make $Q(t)$ interpolate $\hat{D}$, we must find the weight w_i such that $(2w_0/w_1)(Q_1 - Q_0) = U_0$ and $Q'(1) = (2w_2/w_1)(Q_2 - Q_1) = U_1$. Without loss of generality, we can assume $w_0 = 1$. Then w_1 and w_2 can be solved for uniquely as follows. If $Q_1 - Q_0$ and U_0 have the same direction, then $w_1 = 2|Q_1 - Q_0|/|U_0|$; otherwise $w_1 = -2|Q_1 - Q_0|/|U_0|$. Having obtained w_1, if $Q_2 - Q_1$ and w_1U_1 have the same direction, then $w_2 = |w_1U_1|/(2|Q_2 - Q_1|)$; otherwise $w_2 = -|w_1U_1|/(2|Q_2 - Q_1|)$. In cases 2 and 3, and probably in case 4 of Fig. 2, the curve $Q(t)$ is discontinuous, since it contains points at infinity. However, its image curve $P(t) = \mathcal{P}_C(Q(t))$ under stereographic projection is still a continuous SR quartic curve, which interpolates the original data $D = \{X_0, T_0; X_1, T_1\}$, since points at infinity on M_C are mapped by $\mathcal{P}_C$ into well-defined points on S^2. Hence, the existence of SR quartic curves interpolating D is proved.

Remarks. In this proof, when the center of projection C and the projection plane M_C are fixed, the resulting SR quartic curve $P(t)$ interpolating D is unique, since $Q(t)$ is unique on M_C. If we use a different projection plane $\hat{M}_C$, while fixing C, then the resulting SR curve $\hat{P}(t)$ on S^2 interpolating D is the same as $P(t)$, since the intermediate rational quadratic curve $\hat{Q}(t)$ on $\hat{M}_C$ is related to $Q(t)$ under a perspective

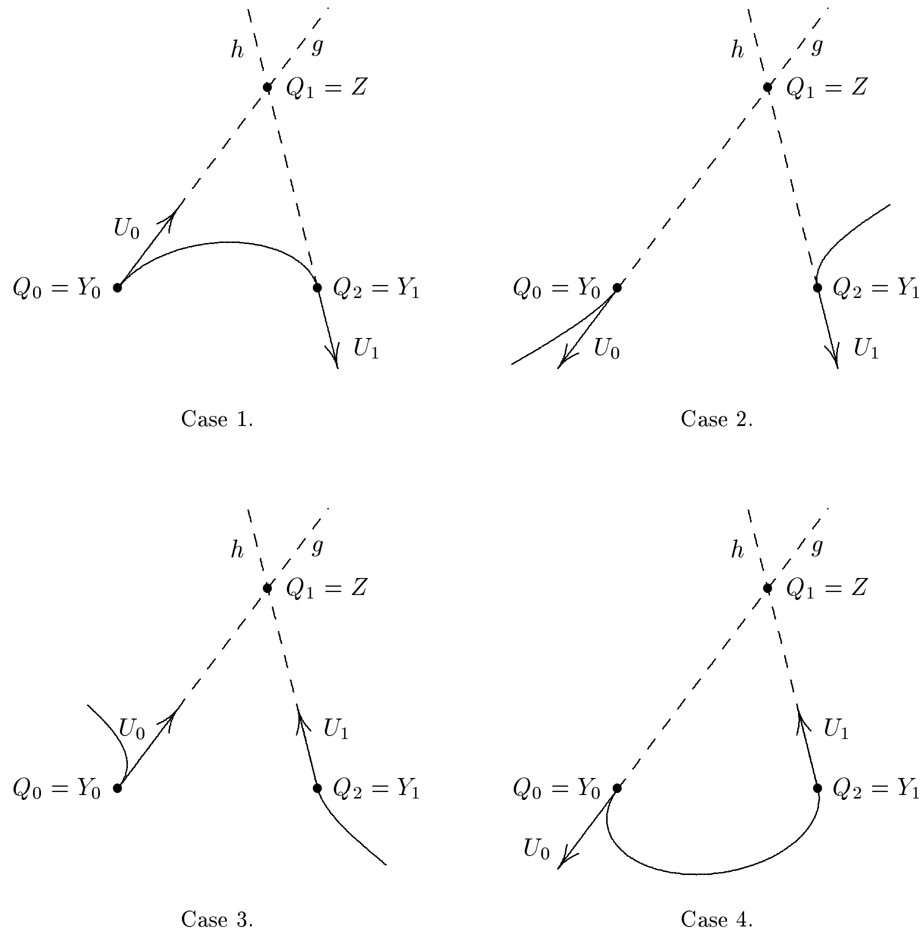

Fig. 2. Four cases of the mapped data

projection. Hence, $P(t)$ is uniquely determined by the center of projection C and is independent of the choice of the projection plane M_C.
By Theorem 1 and the discussion preceding it, we obtain Theorem 2

Theorem 2. *Let X_0 and X_1 be two points on S^n, $n \geq 2$. Let T_0 and T_1 be two nonzero vectors that are tangent to S^n at X_0 and X_1, respectively. There exist SR quartic curves on S^n that interpolate $D = \{X_0, T_0; X_1, T_1\}$.*

We are now interested in knowing how many SR quartic curves there are on S^2 interpolating the given data $D = \{X_0, T_0; X_1, T_1\}$. First, some properties of SR quartic curves on S^2 are given.

Theorem 3. *An SR quartic curve on S^2 has exactly one singular point. Furthermore, an SR quartic curve on S^2 is the image of a rational quadratic curve under the stereographic projection centered at the singular point of the SR quartic curve.*

Remarks. This is implied by a classical result about algebraic curves on a quadric surface [10]. There are two species of quartic curves lying on a quadric surface. A rational quartic curve on a sphere is in the *first species* if it can be obtained as the intersection of two quadrics; a quartic curve in the *second species* is the partial intersection of a quadric and a cubic surface. Here we provide a simple argument for this result.

Proof. Let $P(t)$ be an SR quartic curve on S^2. Let $C = P(t_0)$ be a regular point on $P(t)$. Consider the stereographic projection $\mathcal{P}_C$ with its center at C. It is easy to see that $P(t)$ is mapped by $\mathcal{P}_C^{-1}$ into a rational cubic curve $Q(t)$ in the plane M_C. It is well known that a rational cubic planar curve has exactly one double point [15]. Let $\hat{U}$ denote the double point of $Q(t)$. Let $U = \mathcal{P}_C(\hat{U})$. Then U is a double point of $P(t)$ on S^2. Now use U as the center of another stereographic projection $\mathcal{P}_U$. Then $P(t)$ is mapped by $\mathcal{P}_U^{-1}$ into a rational quadratic curve $\mathcal{P}_U^{-1}(P(t))$ on the projection plane M_U. Hence, $P(t)$ is the image of a rational quadratic curve under a stereographic projection centered at U. This completes the proof.

In the proof, if we construct a quadratic cone with its apex at U and its intersection with the plane M_U being the conic section $\mathcal{P}_U^{-1}(P(t))$, then the curve $P(t)$ plus point U form the intersection between the sphere S^2 and the quadratic cone. Thus, we have Theorem 4.

Theorem 4. *Any SR quartic curve on S^2 is the intersection curve between S^2 and a quadratic cone with its apex on S^2.*

A detailed discussion about the classification of degenerate intersection curves between two quadric surfaces can be found in [2]. The degree of freedom of an SR quartic curve interpolating the given data $D = \{X_0, T_0; X_1, T_1\}$ on S^2 is given by the next theorem.

Theorem 5. *Given the data $D = \{X_0, T_0; X_1, T_1\}$ on S^2, the family of all SR quartic curves on S^2 interpolating $D = \{X_0, T_0; X_1, T_1\}$ has two free parameters.*

Proof. By Theorem 3, all SR quartic curves on S^2 can be obtained as the images of rational quadratic curves through stereographic projection. Given any data $D = \{X_0, T_0; X_1, T_1\}$ on S^2, according to the argument in the proof of Theorem 1, there is a unique SR quartic curve interpolating D for each fixed center of stereographic projection.

Now we just need to show that different points on S^2, when they are used as centers of different stereographic projections, give rise to different SR quartic curves on S^2 interpolating D. Let $P_1(t)$ and $P_2(t)$ be two SR quartic curves interpolating D that are obtained by using two distinct points C_1 and C_2 on S^2 as the centers of stereographic projection, respectively. Then $P_1(t)$ and $P_2(t)$ are two different SR quartic curves, since, by Theorem 3, they have distinct singular points C_1 and C_2. Hence, the degree of freedom of all SR quartic curves on S^2 interpolating D is the same as that of all points on S^2 (except for the points on two circles), which is 2. This completes the proof.

While Theorem 3 states that stereographic projection can be used to generate all SR quartic curves interpolating the given data $D = \{X_0, T_0; X_1, T_1\}$ on S^2, the evidence to be examined indicates that stereographic projection is incapable of generating all SR quartic curves interpolating D if D is given on S^n, where $n \geq 3$. By 'using stereographic projection' we mean here that one aims at obtaining SR quartic curves as the images of rational quadratic curves. Suppose the data $D = \{X_0, T_0; X_1, T_1\}$ on S^n are mapped by a stereographic projection into $\hat{D} = \{Y_0, U_0; Y_1, U_1\}$ to be interpolated by a rational quadratic curve. Since any rational quadratic curve is necessarily planar, the data $\hat{D}$ must be contained in a 2-D plane. However, for general data $D = \{X_0, T_0; X_1, T_1\}$ on S^n, it is easy to see that $\hat{D}$ is contained in a 2-D plane if and only if the center of projection is on the 2-D sphere S that is the intersection between S^n and the unique 3-D flat H spanned by D. By an argument similar to that leading to Theorem 5, we conclude that the family of SR quartic curves on S^n interpolating D that can be obtained by the stereographic projection approach has only two degrees of freedom, and all these SR quartic curves lie on the 2-D sphere S, hence in the 3-D flat H. As a rational quartic curve naturally spans a 4D space, we suspect that the SR quartic curves given by stereographic projection form only a subset of all possible SR quartic curves on S^n, $n \geq 3$. Indeed, in the next section we take an algebraic approach to generating all SR quartic curves interpolating the given data $D = \{X_0, T_0; X_1, T_1\}$ on S^n, and show that the family of these curves actually has n degrees of freedom.

3 Computation

While stereographic projection is used in the existence proof, we recognize three problems with using it for computing SR quartic interpolating curves.

1. Such a construction scheme would depend on first choosing a center of projection, which is a difficult task unless the data D is well behaved,

i.e. X_0 and X_1 are close to each other and the directions of T_0 and T_1 do not deviate much from the direction of vector $X_1 - X_0$.
2. The stereographic projection is not distance preserving, and there is in general considerable shape distortion between the intermediate interpolating rational quadratic curve $Q(t)$ and its image $P(t) = \mathcal{P}_C(Q(t))$, especially when discontinuous curves $Q(t)$ are encountered, as in cases 2 and 3 in Fig. 2.
3. Most importantly, as suggested at the end of last section and to be verified later in this section, the stereographic projection images of rational quadratic curves do not yield all SR quartic curves on S^n.

Based on these considerations, we shall study a direct algebraic approach to computing SR quartic curves on S^n interpolating the data $D = \{X_0, T_0; X_1, T_1\}$, given the existence of such curves by Theorem 2.

In the following, a point X is represented by homogeneous coordinates $X = (x_0, x_1, \ldots, x_n, w)^T$ in E^{n+1}. For a finite point X with $w \neq 0$, we call $(x_0/w, x_1/w, \ldots, x_n/w, 1)^T$ the *standard form* of X.

Consider a rational quartic curve in homogeneous coordinates in Bézier form

$$P(t) = \varrho_0 P_0 B_{0,4}(t) + w_0 P_1 B_{1,4}(t) + P_2 B_{2,4}(t) + w_1 P_3 B_{3,4}(t) + \varrho_1 P_4 B_{4,4}(t), \quad t \in [0, 1].$$

We assume that all the P_i, except for P_2, are in the standard form. $P(t)$ is used to interpolate data points X_0, X_1 on S^n and end tangent vectors T_0 and T_1 specified at X_0 and X_1, respectively. Here the X_i, $i = 0, 1$, are in the form $X_i = (x_{0,i}, x_{1,i}, \ldots, x_{n,i}, 1)^T$ and the T_i, $i = 0, 1$, are in the form $T_i = (t_{0,i}, t_{1,i}, \ldots, t_{n,i}, 0)^T$.

Denote the standard form of $P(t)$ by $\tilde{P}(t)$. Then the interpolation conditions are

$$\tilde{P}(0) = X_0, \quad \tilde{P}(1) = X_1$$
$$\tilde{P}'(0) = T_0, \quad \tilde{P}'(1) = T_1.$$

It follows first that $P_0 = X_0$ and $P_4 = X_1$. It is easy to verify that

$$\tilde{P}'(0) = \frac{4w_0}{\varrho_0}(P_1 - P_0).$$

Then it follows from $\tilde{P}'(0) = T_0$ that

$$P_1 = X_0 + \frac{\varrho_0}{4w_0} T_0. \tag{1}$$

Similarly,

$$P_3 = X_1 - \frac{\varrho_1}{4w_1} T_1. \tag{2}$$

Setting $V_0 = T_0/4$ and $V_1 = -T_1/4$, we obtain

$$P_1 = X_0 + \frac{\varrho_0}{w_0} V_0, \qquad P_3 = X_1 + \frac{\varrho_1}{w_1} V_1.$$

Thus, $P(t)$ can be written as

$$P(t) = \varrho_0 X_0 B_{0,4}(t) + (w_0 X_0 + \varrho_0 V_0) B_{1,4}(t) + P_2 B_{2,4}(t) + (w_1 X_1 + \varrho_1 V_1) B_{3,4}(t) + \varrho_1 X_1 B_{4,4}(t), \quad t \in [0, 1].$$

Let S^n be represented by $X^T AX = 0$, where $A = \mathrm{diag}[1, 1, \ldots, 1, -1]$ is an $(n+2) \times (n+2)$ matrix. Then $P(t)^T AP(t) = 0$ for all t. Using the relation

$$B_{i,4}(t) B_{j,4}(t) = \frac{4!4!(i+j)!(8-i-j)!}{8!i!(4-i)!j!(4-j)!} B_{i+j,8}(t),$$

$P(t)^T AP(t)$ can be expressed as a linear combination of the basis functions $B_{k,8}(t)$, $0 \le k \le 8$. Since $P(t)^T AP(t) = 0$, all the nine coefficients of the $B_{k,8}(t)$ in this expression should be zero. Since

$$X_0^T AX_0 = X_0^T AV_0 = X_1^T AX_1 = X_1^T AV_1 = 0, \tag{3}$$

the first two and the last two coefficients vanish automatically. The vanishing of the five remaining coefficients leads to the equations

$$\frac{3}{7}\varrho_0 X_0^T AP_2 + \frac{4}{7}(w_0 X_0 + \varrho_0 V_0)^T \times A(w_0 X_0 + \varrho_0 V_0) = 0 \tag{4}$$

$$\frac{1}{7}\varrho_0 X_0^T A(w_1 X_1 + \varrho_1 V_1) + \frac{6}{7}(w_0 X_0 + \varrho_0 V_0)^T \times AP_2 = 0 \tag{5}$$

$$\frac{18}{35} P_2^T AP_2 + \frac{16}{35}(w_0 X_0 + \varrho_0 V_0)^T A(w_1 X_1 + \varrho_1 V_1) + \frac{1}{35}\varrho_0 \varrho_1 X_0^T AX_1 = 0 \tag{6}$$

$$\frac{1}{7}\varrho_1 X_1^T A(w_0 X_0 + \varrho_0 V_0) + \frac{6}{7}(w_1 X_1 + \varrho_1 V_1)^T \times AP_2 = 0 \tag{7}$$

$$\frac{3}{7}\varrho_1 X_1^T AP_2 + \frac{4}{7}(w_1 X_1 + \varrho_1 V_1)^T \times A(w_1 X_1 + \varrho_1 V_1) = 0. \tag{8}$$

Using (3) and assuming $\varrho_0\varrho_1 \neq 0$, these equations can be simplified into

$$3X_0^T AP_2 + 4\varrho_0 V_0^T AV_0 = 0 \tag{9}$$

$$\varrho_0 X_0^T A(w_1X_1 + \varrho_1 V_1) + 6(w_0X_0 + \varrho_0 V_0)^T \times AP_2 = 0 \tag{10}$$

$$18P_2^T AP_2 + 16(w_0X_0 + \varrho_0V_0)^T A(w_1X_1 + \varrho_1V_1) + \varrho_0\varrho_1 X_0^T AX_1 = 0 \tag{11}$$

$$\varrho_1 X_1^T A(w_0X_0 + \varrho_0V_0) + 6(w_1X_1 + \varrho_1V_1)^T \times AP_2 = 0 \tag{12}$$

$$3X_1^T AP_2 + 4\varrho_1 V_1^T AV_1 = 0. \tag{13}$$

From Eqs. 9 and 13, there are

$$X_0^T AP_2 = -\frac{4}{3}\varrho_0 V_0^T AV_0 \quad \text{and}$$
$$X_1^T AP_2 = -\frac{4}{3}\varrho_1 V_1^T AV_1. \tag{14}$$

Substituting them into Eqs. 10 and 12 respectively, removing the factors ϱ_0 and ϱ_1, and rearranging the order, we obtain the following of equations

$$3X_0^T AP_2 = -4\varrho_0 V_0^T AV_0 \tag{15}$$

$$3X_1^T AP_2 = -4\varrho_1 V_1^T AV_1 \tag{16}$$

$$6V_0^T AP_2 = 8(V_0^T AV_0)w_0 - (X_0^T AX_1)w_1 - (X_0^T AV_1)\varrho_1 \tag{17}$$

$$6V_1^T AP_2 = -(X_0^T AX_1)w_0 + 8(V_1^T AV_1)w_1 - (X_1^T AV_0)\varrho_0 \tag{18}$$

$$18P_2^T AP_2 + 16(w_0X_0 + \varrho_0V_0)^T A(w_1X_1 + \varrho_1V_1) + \varrho_0\varrho_1 X_0^T AX_1 = 0. \tag{19}$$

Here the last equation is identical to Eq. 11. This is a system of five homogeneous equations binding $n+6$ homogeneous variables ($n+2$ variable coordinates of P_2 plus the four weights ϱ_0, ϱ_1, w_0, and w_1). Thus, in general, the number of independent parameters is $(n+6)-5-1 = n$. By Theorem 2, these equations must be consistent and have real solutions.

Now we discuss how to solve this system of equations. The general idea is to substitute the weights ϱ_0, ϱ_1, w_0, and w_1 in Eq. 19 to turn it into a quadratic equation in P_2. Let $\Delta \equiv 64(V_0^T AV_0)(V_1^T AV_1) - (X_0^T AX_1)^2$. There are two cases to consider: (1) $\Delta \neq 0$ and (2) $\Delta = 0$.

In case 1, using the relations of Eq. 14, it follows from Eq. 15 – 18 that

$$\varrho_0 = -\frac{3X_0^T AP_2}{4V_0^T AV_0} \tag{20}$$

$$\varrho_1 = -\frac{3X_1^T AP_2}{4V_1^T AV_1} \tag{21}$$

$$8(V_0^T AV_0)w_0 - (X_0^T AX_1)w_1 = 6V_0^T AP_2 - \frac{3X_0^T AV_1}{4V_1^T AV_1}(X_1^T AP_2) \tag{22}$$

$$-(X_0^T AX_1)w_0 + 8(V_1^T AV_1)w_1 = 6V_1^T AP_2 - \frac{3X_1^T AV_0}{4V_0^T AV_0}(X_0^T AP_2). \tag{23}$$

Since $\Delta \neq 0$, w_0 and w_1 can be expressed linearly in terms of P_2 from the last two equations. Substituting w_0, w_1, ϱ_0, and ϱ_1 into Eq. 19, we obtain a homogeneous quadratic equation, denoted by $F_1(P_2) = 0$, in the $n+2$ coordinates of P_2. Each solution P_2 of $F_1(P_2) = 0$ determines uniquely the values of w_0, w_1, ϱ_0, and ϱ_1, which in turn yield an SR quartic curve interpolating $D = \{X_0, T_0; X_1, T_1\}$.

By Theorem 2, $F_1(P_2) = 0$ has real solutions. For better notation, we denote the equation $F_1(P_2) = 0$ by $F_1(Y) = 0$, with Y standing for the $n+2$ variable coordinates of P_2. The standard way to find all real points on the quadric surface $F_1(Y) = 0$ is to reduce $F_1(Y) = 0$ by affine transformation into a canonical form $\tilde{F}_1(\tilde{X}) = 0$. It is then easy to find a real point $\tilde{C}_1$ on surface $\tilde{F}_1(\tilde{Y}) = 0$, and therefore a corresponding real point C_1 on $F_1(Y) = 0$. Using C_1 as a center of projection, a rational quadratic parameterization of the quadric surface $F_1(Y) = 0$ can be obtained [12]. This parameterization gives out all real points P_2 on $F_1(Y) = 0$, except for the center C_1.

In case 2, since $\Delta = 0$, w_0 and w_1 cannot be isolated from Eqs. 22 and 23. In this case, for Eqs. 22 and 23 to be consistent, the following linear condition must be imposed on P_2.

$$\begin{vmatrix} 8(V_0^T AV_0) & 6V_0^T AP_2 - \frac{3X_0^T AV_1}{4V_1^T AV_1}(X_1^T AP_2) \\ -(X_0^T AX_1) & 6V_1^T AP_2 - \frac{3X_1^T AV_0}{4V_0^T AV_0}(X_0^T AP_2) \end{vmatrix} = 0, \tag{24}$$

which is denoted by $L_2(P_2) = 0$. Since the system of equations under consideration is homogeneous, we may set $w_0 = 1$. Then w_1 can be solved for from Eq. 22 as

$$w_1 = \frac{1}{X_0^T AX_1}\left[8(V_0^T AV_0) - 6V_0^T AP_2 + \frac{3X_0^T AV_1}{4V_1^T AV_1}(X_1^T AP_2)\right].$$

Setting $w_0 = 1$, substituting this w_1, and ϱ_0 and ϱ_1 from Eqs. 20 and 21, into Eq. 19, we obtain an inhomogeneous quadratic equation in P_2, denoted by $F_2(P_2) = 0$. Thus, P_2 is determined by $L_2(P_2) = 0$ and $F_2(P_2) = 0$.

Again, by Theorem 2, there are real solutions P_2 satisfying $L_2(Y) = 0$ and $F_2(Y) = 0$; here Y denotes the coordinates of P_2. All real solutions P_2 can be found by the following procedure. First we pick $n+1$ linearly independent points U_i, $i = 0, 1, \ldots, n$, on the hyperplane $L_2(Y) = 0$. Then we obtain a linear parameterization of $L_2(Y) = 0$ such as

$$Y(R) = r_0U_0 + r_1U_1 + \cdots + r_nU_n,$$

where $R = (r_0, r_1, \ldots, r_n)$. Substituting $Y(R)$ into $F_2(Y) = 0$, we obtain a quadric surface $G_2(R) \equiv F_2(Y(R)) = 0$, which is inhomogeneous, since $F_2(Y) = 0$ is inhomogeneous. Then the similar procedure in case 1 can be used to parameterize $G_2(R) = 0$ to get all the real points on $G_2(R) = 0$. These points in turn give out all solutions P_2 through $P_2 = Y(R)$.

Clearly, in either case 1 or case 2, there are n independent free parameters in the solution of P_2. Hence, the family of all SR quartic curves interpolating $D = \{X_0, T_0; X_1, T_1\}$ on S^n has n degrees of freedom.

Now we use a running example to illustrate the process of computing an SR quartic interpolating curve. Consider $D = \{X_0, T_0; X_1, T_1\}$ on S^2, where $X_0 = (1, 0, 0, 1)^T$, $T_0 = (0, 1, 0, 0)^T$, $X_1 = (0, 1, 0, 1)^T$, and $T_1 = (0, 0, 1, 0)^T$. Then $V_0 = (0, 1/4, 0, 0)^T$ and $V_1 = (0, 0, -1/4, 0)^T$. Since $\Delta = -3/4 \neq 0$, we have case 1 at hand, and we can solve for w_0 and w_1 from Eqs. 22 and 23. We then obtain

$$\varrho_0 = (-12, 0, 0, 12)P_2 \quad (25)$$

$$\varrho_1 = (0, -12, 0, 12)P_2 \quad (26)$$

$$w_0 = (-4, -1, -2, 4)P_2 \quad (27)$$

$$w_1 = (2, 2, 1, -2)P_2. \quad (28)$$

Substituting these into Eq. 19, we have a quadratic equation in P_2, denoted by $Y^T MY = 0$, where

$$M = \begin{bmatrix} 5 & -4 & 4 & 4 \\ -4 & 5 & 4 & 4 \\ 4 & 4 & 5 & -4 \\ 4 & 4 & -4 & -13 \end{bmatrix}.$$

It is easy to see that $P_2 = (0, 0, -1, 1)^T$ is a solution of $Y^T MY = 0$. Using this P_2, by Eqs. 25 – 28, we find the weights ϱ_0, ϱ_1, w_0, and w_1, which in turn yield the control points P_1 and P_3 through Eq. 1 and 2. Finally, we obtain the following SR quartic curve interpolating D.

$$P(t) = \varrho_0 P_0 B_{0,4}(t) + w_0 P_1 B_{1,4}(t) + P_2 B_{2,4}(t) + w_1 P_3 B_{3,4}(t) + \varrho_1 P_4 B_{4,4}(t), \quad t \in [0, 1],$$

where $\varrho_0 = \varrho_1 = 12$, $w_0 = 6$, $w_1 = -3$, $P_0 = (1, 0, 0, 1)^T$, $P_1 = (1, 0.5, 0, 1)^T$, $P_2 = (0, 0, -1, 1)^T$, $P_3 = (0, 1, 1, 1)^T$, and $P_4 = (0, 1, 0, 1)^T$.

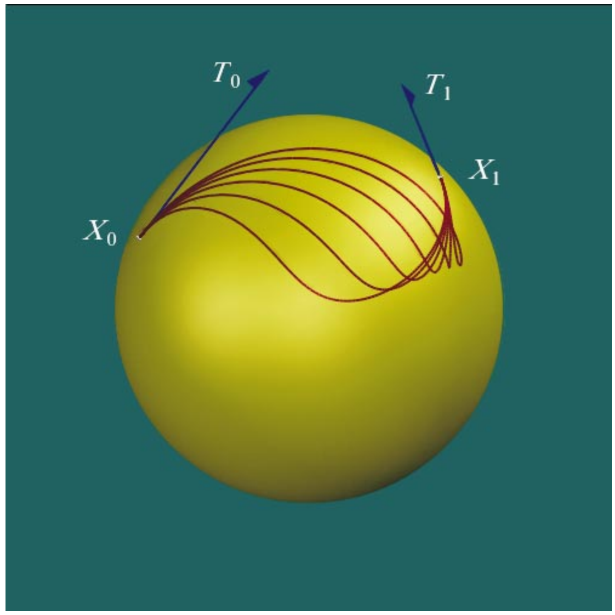

Fig. 3. Some SR quartic interpolating curves on S^2

Figure 3 shows another example of some SR quartic curves on S^2 interpolating the data $D = \{X_0, T_0; X_1, T_1\}$, with $X_0 = (1, 0, 0, 1)^T$, $T_0 = (0, 3, 0, 0)^T$, $X_1 = (0, 1, 0, 1)^T$, and $T_1 = (0, 0, 1, 0)^T$.

Figure 4 shows a spherical motion generated by an SR quartic curve on S^3 interpolating the data $D = \{X_0, T_0; X_1, T_1\}$, with $X_0 = (0, 0, 0, 1, 1)^T$, $T_0 = (0, 1, 0, 0, 0)^T$, $X_1 = (0, 1, 0, 0, 1)^T$, and $T_1 = (0, 0, 1, 0, 0)^T$; a point $(x_0, x_1, x_2, x_3, 1)^T \in S^3$ is

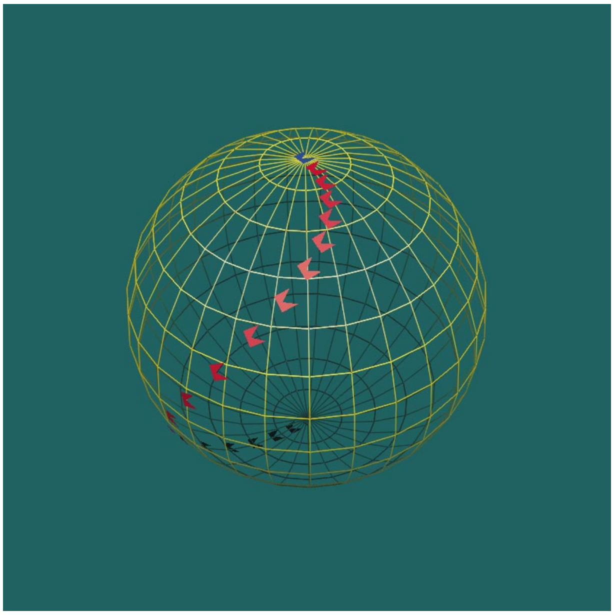

Fig. 4. The spherical motion generated by a 4D SR curve

identified with the unit quaternion $q = x_3 + x_0\boldsymbol{i} + x_1\boldsymbol{j} + x_2\boldsymbol{k}$. Here $P_2 \approx (2.687419, 0.0, 0.0, 0.0, 1.0)^T$. Clearly, $P(t)$ is a 4-D curve since P_2 is not contained in the 3-space spanned by X_0, X_1, T_0, and T_1.

4 Conclusion

We have shown that there exist SR quartic curves interpolating any Hermite data $D = \{X_0, T_0; X_1, T_1\}$ on S^n, $n \geq 2$, and all these curves form a family with n degrees of freedom. In addition, it is shown that, except on S^2, not all of these curves can be generated as the images of rational quadratic curves under stereographic projection, which is the main approach used in many existing methods in the literature for constructing SR curves. We also present an algebraic method of computing all SR quartic curves interpolating data $D = \{X_0, T_0; X_1, T_1\}$ on S^n. An interesting note is that any SR quartic curve on S^2 is the intersection curve between S^2 and a unique quadratic cone with its apex on S^2.

From the viewpoint of CAGD, given the considerable degree of freedom of SR quartic curves on S^n, an important problem for further research is to study the shape property and velocity control of these interpolating curves.

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